01/31/2023 Dimensionality Reduction (PCA) Continued

- ~~orthogonal columns are perpendicular dot product gives 0, rectangular, and unit length, orthonormal are square orthogonal matrices~~ at end of notes
  - example

$$\begin{bmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

orthogonal — orthonormal

- terms to know for Quiz 1 (02/02/2023)
  - rank of a matrix
  - linearly dependent/independent matrix
  - column space/row space of a matrix
  - null space of a matrix
  - diagonal/symmetric/rank deficient matrix
  - solving linear system (elementary row operations)
  - 3 applications
    - Image Compression
    - Image Encoding
    - Dimensionality Reduction using PCA
- PCA

$$A = \begin{bmatrix} — a_1 — \\ — a_2 — \\ \vdots \\ — a_n — \end{bmatrix}_{n \times d}$$

- procedure

1.) Center the data

$$B = \begin{bmatrix} - & a_1 - \bar{a} & - \\ - & a_2 - \bar{a} & - \\ & \vdots & \\ - & a_n - \bar{a} & - \end{bmatrix}$$

2.) Covariance Matrix

$$C = B^T B$$

3.) Eigendecomposition of C

$$C = V \Sigma V^T$$

4.) Principal Components

$$T = BV$$

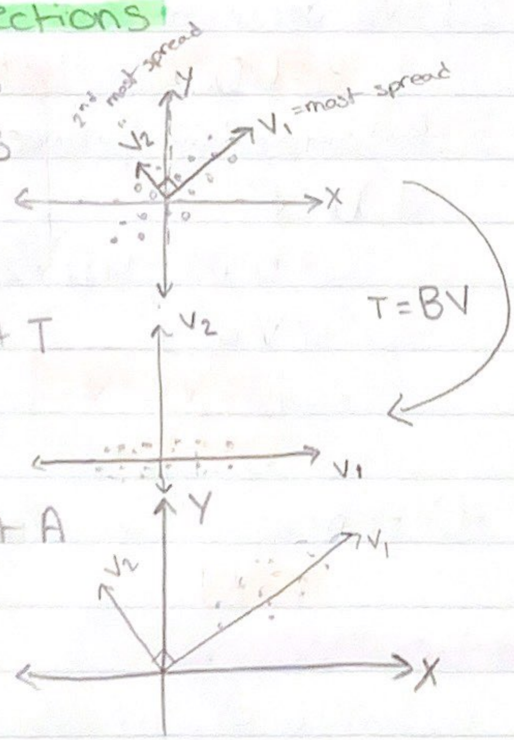
V: loading matrix/principal directions

example:

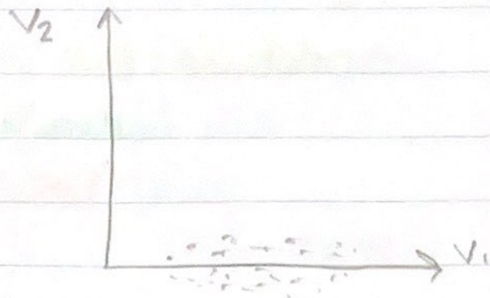
Dataset B

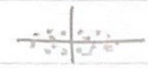
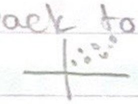
Dataset T

Dataset A



Dataset T
if $T = AV$
instead of
$T = BV$



- not centering is MATHEMATICALLY correct, but not NUMERICALLY stable
  - ★★ $\boxed{\text{ALWAYS CENTER}}$ ★★
- A → center A (B) → V → Project B on V → 
  → going back to A →  → add translation ($\bar{a}$) → 

- Relationship between SVD and PCA
  1.) Start with centered matrix B
  2.) SVD decomposition of B

  $B = UDV^T$ (let Σ of SVD be D)

  3.) Relationship between C ($C = B^TB$) and U, D, V^T

  $$C = B^TB = (UDV^T)^T(UDV^T) = VD\underbrace{U^TU}_{I}DV^T$$
  $$= VD^2V^T$$
  $$\therefore \boxed{C = VD^2V^T}$$
  $$\therefore D^2 = \Sigma \leftarrow \text{from } C = V\Sigma V^T$$

  4.) Principal Components

  $T = BV$

- important notes
  1.) Square of singular values of B are same

as eigenvalues of C

2.) Right singular vectors of B are same as eigenvectors of C

- Orthonormal Properties / Conditions
  - square
  - unit length / normal columns
  - columns are perpendicular (i.e., dot product is 0)
  - if orthonormal matrix is called A, then $A^T A = I$ and AA^T = Diagonal Matrix
- Unnormalized orthonormal matrices are orthogonal