01/24/2023 SVD Continued

$$A = \begin{bmatrix} | & | & | & | \\ a_1 & a_2 & a_3 & a_4 \\ | & | & | & | \end{bmatrix} = \underset{100\times100}{U} \overbrace{\underset{100\times4}{\Sigma} \underset{4\times4}{V^T}}^{W} = \underset{100\times100}{U} \underset{100\times4}{W}$$

$$\underset{100\times4}{W} = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_4 \\ & & 0 & \\ & & \vdots & \\ & & 0 & \end{bmatrix} \begin{bmatrix} — V_1^T — \\ — V_2^T — \\ — V_3^T — \\ — V_4^T — \end{bmatrix} = \begin{bmatrix} — \sigma_1 V_1^T — \\ — \sigma_2 V_2^T — \\ — \sigma_3 V_3^T — \\ — \sigma_4 V_4^T — \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Since all rows after the 4^{th} row (rank) is 0, we will only keep the first 4 rows in W and first 4 columns in U

$$\underset{100\times4}{A = U} \underset{4\times4}{W} = \begin{bmatrix} | & | & | & | \\ u_1 & u_2 & u_3 & u_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} W_{11} & W_{21} & W_{31} & W_{41} \\ W_{12} & W_{22} & W_{32} & W_{42} \\ W_{13} & W_{23} & W_{33} & W_{43} \\ W_{14} & W_{24} & W_{34} & W_{44} \end{bmatrix}$$

$$a_1 = \sum_{i=1}^{4} u_i W_{1i}$$

- Can you represent a new sample or column of A (a_5 in this case) using the original columns in A (a_1, a_2, a_3, a_4 in this case)?

$$a_5 = \sum_{i=1}^{4} u_i \alpha_i$$

- we need to solve an optimization problem (least squares)

our problem	Linear Regression
$\min \| a_5 - \sum_{i=1}^{4} u_i \alpha_i \|_2^2$	$y = \beta_0 + \beta_1 X$
$\alpha = [\alpha_1, \alpha_2, \alpha_3, \alpha_4]$	$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$
let $\underset{100\times 4}{U} = \mathcal{L}$	$\min \| Y - X\beta \|_2^2$
$\min \| a_5 - \mathcal{L}\alpha \|_2^2$	$\beta = [\beta_0, \beta_1]$
$\alpha = [\alpha_1, \alpha_2, \alpha_3, \alpha_4]$	$\beta = (X^T X)^{-1} X^T Y$
$\alpha = (L^T L)^{-1} L^T a_5$	
but L is orthogonal!	
$\therefore L^T L = I$	
$\alpha = L^T a_5$	

- approximation of a_5: $\overline{a_5} = L\alpha = LL^T a_5$

** - LL^T is not always I, but $L^T L$ is always I **

- go through 2_SVD_Eigenbases_faces.ipynb