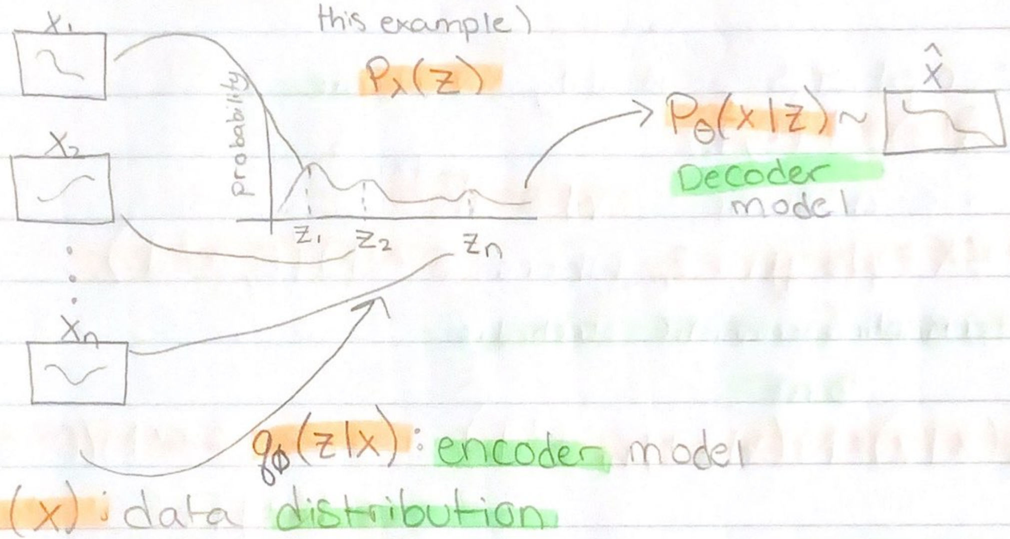
Generative Models (Probabilistic PCA) 02/21/2023

- general idea

$x_1, x_2, \ldots x_n \sim P_{\theta}x$ (1 dimension in this example)



$P_\theta(x)$: data distribution

$P_\lambda(z)$: distribution of latent representation where λ is a set of parameters

$P_\theta(x|z)$: conditional distribution of $x|z$ where θ is a set of parameters

$q_\phi(z|x)$: conditional distribution of $z|x$ where ϕ is a set of parameters

- marginal distribution

$$P_\theta(x) = \int P_\theta(x,z)\,dz = \int P_\theta(x|z)\,P_\lambda(z)\,dz$$

$$\frac{P(x,z)}{P(z)} = P(x|z) \Rightarrow \int P(x,z)\,dz = \int P(x|z)\,P(z)\,dz$$

- we optimize λ and θ such that likelihood of the dataset is maximized

$$\max_\theta \log P_\theta(x) = \max_\theta \log \int P_\theta(x|z)\,P_\lambda(z)\,dz$$

- if θ is optimized, then the chance of getting back the original image is high

- assumptions

identity covariance

$P(z) = N(0, I)$

we assume $W \in \mathbb{R}^{d \times m}$

$P_\theta(x|z): X = \underbrace{Wz}_{\text{deterministic}} + \underbrace{\mu}_{\substack{\text{deterministic} \\ \text{bias}}} + \underbrace{\epsilon}_{\text{stochatic}}$ where $\epsilon \sim N(0, \sigma^2 I)$

$P_\theta(x|z) = N(Wz + \mu, \sigma^2 I)$

since $\epsilon \sim N(0, \sigma^2 I)$
$Wz + \mu + \epsilon \sim (Wz + \mu, \sigma^2 I)$

$$\left.\begin{aligned} z &\sim P(z) = N(0, I) \\ x &\sim P_\theta(x|z) = N(Wz + \mu, \sigma^2 I) \end{aligned}\right\} \text{generative process}$$

parameters: $[W, \mu, \sigma]$

- side note

$P(z) = N(\alpha, \Lambda^{-1})$

$P(x|z) = N(Az + b, L^{-1})$

$P(x) \int P(x|z) P(z) dz = N(A\alpha + b, L^{-1} + A\Lambda^{-1}A^T)$

$P(z|x) = \dfrac{P(x,z)}{P(x)} = N(\Sigma(A^T L(x-b) + \Lambda\alpha), \Sigma)$

where $\Sigma = (\Lambda + A^T L A)^{-1}$

- with our assumptions

$W = A$, $z = z$, $\mu = b$, $\sigma^2 I = L^{-1}$, $I = \Lambda^{-1}$, $0 = \alpha$

$\therefore P_\theta(x) = N(\mu, \sigma^2 I + WW^T)$ and $\Sigma(I + W^T(\frac{1}{\sigma^2} I)W)^{-1}$